

**Opening the Crystalline Color Superconductivity Window**

Adam K. Leibovich*

*Theory Group, Fermilab**P.O. Box 500, Batavia, IL 60510*Krishna Rajagopal[†], Eugene Shuster[‡]*Center for Theoretical Physics**Massachusetts Institute of Technology**Cambridge, MA 02139*

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Abstract

Cold dense quark matter is in a crystalline color superconducting phase wherever pairing occurs between species of quarks with chemical potentials whose difference $\delta\mu$ lies within an appropriate window. If the interaction between quarks is modeled as point-like, this window is rather narrow. We show that when the interaction between quarks is modeled as single-gluon exchange, the window widens by about a factor of ten at accessible densities and by much larger factors at higher density. This striking enhancement reflects the increasingly $(1+1)$ -dimensional nature of the physics at weaker and weaker coupling. Our results indicate that crystalline color superconductivity is a generic feature of the phase diagram of cold dense quark matter, occurring wherever one finds quark matter which is not in the color-flavor locked phase. If it occurs within the cores of compact stars, a crystalline color superconducting region may provide a new locus for glitch phenomena.

*Email address: adam@fnal.gov

[†]Email address: krishna@ctp.mit.edu

[‡]Email address: eugeneus@mit.edu

I. INTRODUCTION

At asymptotic densities, the ground state of QCD with three quarks with equal masses is expected to be the color-flavor locked (CFL) phase [1–4]. This phase features a condensate of Cooper pairs of quarks which includes ud , us , and ds pairs. Quarks of all colors and all flavors participate equally in the pairing, and all excitations with quark quantum numbers are gapped.

The CFL phase persists for unequal quark masses, so long as the differences are not too large [5,6]. In the absence of any interaction (and thus in the absence of CFL pairing) a quark mass difference pushes the Fermi momenta for different flavors apart, yielding different number densities for different flavors. In the CFL phase, however, the fact that the pairing energy is maximized when u , d , and s number densities are equal enforces this equality [7]. This means that if one imagines increasing the strange quark mass m_s , all quark number densities remain equal until a first order phase transition, at which CFL pairing is disrupted, (some) quark number densities spring free under the accumulated tension, and a less symmetric state of quark matter is obtained [7]. This state of matter, which results when pairing between two species of quarks persists even once their Fermi momenta differ, is a crystalline color superconductor.

We can study crystalline color superconductivity more simply by focusing just on pairing between massless up and down quarks whose Fermi momenta we attempt to push apart by turning on a chemical potential difference, rather than a quark mass difference. That is, we introduce

$$\begin{aligned}\mu_u &= \bar{\mu} - \delta\mu \\ \mu_d &= \bar{\mu} + \delta\mu .\end{aligned}\tag{1}$$

If $\delta\mu$ is nonzero but less than some $\delta\mu_1$, the ground state is precisely that obtained for $\delta\mu = 0$ [8–10]. In this state, red and green up and down quarks pair, yielding four quasiparticles with superconducting gap Δ_0 [11–14]. Furthermore, the number density of red and green up quarks is the same as that of red and green down quarks. As long as $\delta\mu$ is not too large, this BCS state remains unchanged (and favored) because maintaining equal number densities, and thus coincident Fermi surfaces, maximizes the pairing and hence the gain in interaction energy. As $\delta\mu$ is increased, the BCS state remains the ground state of the system only as long as its negative interaction energy offsets the large positive free energy cost associated with forcing the Fermi seas to deviate from their normal state distributions. In the weak coupling limit, in which $\Delta_0/\bar{\mu} \ll 1$, the BCS state persists for $\delta\mu < \delta\mu_1 = \Delta_0/\sqrt{2}$ [8,10].

These conclusions are the same (as long as $\Delta_0/\bar{\mu} \ll 1$) whether the interaction between quarks is modeled as a point-like four-fermion interaction or is approximated by single-gluon exchange [3]. The latter analysis [15–22,3,4] is of quantitative validity at densities which are so extremely high that the QCD coupling $g(\mu) < 1$ [23]. Applying these asymptotic results at accessible densities nevertheless yields predictions for the magnitude of the BCS gap Δ_0 , and thus for $\delta\mu_1$, which are in qualitative agreement with those made using phenomenologically normalized models with simpler point-like interactions [19,3,4].

Above the BCS state, in a range $\delta\mu_1 < \delta\mu < \delta\mu_2$, the crystalline color superconducting phase occurs. We shall demonstrate in this paper that applying asymptotic results at accessible densities yields a window $(\delta\mu_2 - \delta\mu_1)$ that is more than a factor of ten wider than

in models with point-like interactions. The crystalline color superconductivity window in parameter space may therefore be much wider than previously thought, making this phase a generic feature of the phase diagram for cold dense quark matter. The reason for this qualitative increase in $\delta\mu_2$ can be traced back to the fact that gluon exchange at weaker and weaker coupling is more and more dominated by forward-scattering, while point-like interactions describe s -wave scattering. What is perhaps surprising is that even at quite *large* values of g , gluon exchange yields an order of magnitude increase in $\delta\mu_2 - \delta\mu_1$.

The crystalline color superconducting state is the analogue of a state first explored by Larkin and Ovchinnikov [24] and Fulde and Ferrell [25] in the context of electron superconductivity in the presence of magnetic impurities. Translating LOFF's results to the case of interest, the authors of Ref. [10] found that for $\delta\mu \gtrsim \delta\mu_1$ it is favorable to form a state in which the u and d Fermi momenta are given by μ_u and μ_d as in the absence of interactions, and are thus not equal, but pairing nevertheless occurs. Whereas in the BCS state, obtained for $\delta\mu < \delta\mu_1$, pairing occurs between quarks with equal and opposite momenta, when $\delta\mu \gtrsim \delta\mu_1$ it is favorable to form a condensate of Cooper pairs with nonzero total momentum. This is favored because pairing quarks with momenta which are not equal and opposite gives rise to a region of phase space where each quark in a Cooper pair can be close to its own Fermi surface, even when the up and down Fermi momenta differ, and such pairs can be created at low cost in free energy.¹ Condensates of this sort spontaneously break translational and rotational invariance, leading to gaps which vary periodically in a crystalline pattern. If in some shell within the quark matter core of a neutron star (or within a strange quark star) the quark chemical potentials are such that crystalline color superconductivity arises, as we now see occurs for a wide range of reasonable parameter values, rotational vortices may be pinned in this shell, making it a locus for glitch formation [10]. Rough estimates of the pinning force suggest that it is comparable to that for a rotational vortex pinned in the inner crust of a conventional neutron star, and thus may yield glitches of phenomenological interest [10].

As in Refs. [10,29], we shall restrict our attention here to the simplest possible “crystal” structure, namely that in which the condensate varies like a plane wave:

$$\langle\psi(\mathbf{x})\psi(\mathbf{x})\rangle \propto \Delta e^{2i\mathbf{q}\cdot\mathbf{x}} . \quad (2)$$

Wherever this condensate is favored over the homogeneous BCS condensate and over the state with no pairing at all, we expect that the true ground state of the system is a condensate which varies in space with some more complicated spatial dependence. The phonon associated with the plane wave condensate (2) has recently been analyzed [30]. A full analysis of

¹LOFF condensates have also recently been considered in two other contexts. In QCD with $\mu_u < 0$, $\mu_d > 0$ and $\mu_u = -\mu_d$, one has equal Fermi momenta for \bar{u} antiquarks and d quarks, BCS pairing occurs, and consequently a $\langle\bar{u}d\rangle$ condensate forms [26,27]. If $-\mu_u$ and μ_d differ, and if the difference lies in the appropriate range, a LOFF phase with a spatially varying $\langle\bar{u}d\rangle$ condensate results [26,27]. Our conclusion that the LOFF window is much wider than previously thought applies in this context also. Suitably isospin asymmetric nuclear matter may also admit LOFF pairing, as discussed recently in Ref. [28].

all the phonons in the crystalline color superconducting phase must await the determination of the actual crystal structure.

The authors of Refs. [10,29] studied crystalline color superconductivity in a model in which two flavors of quarks with chemical potentials (1) interact via a point-like four-fermion interaction with the quantum numbers of single gluon exchange. In the LOFF state, each Cooper pair has total momentum $2\mathbf{q}$ with $|\mathbf{q}| \approx 1.2\delta\mu$ [10]. The direction of \mathbf{q} is chosen spontaneously. The LOFF phase is characterized by a gap parameter Δ and a diquark condensate, but not by an energy gap: the quasiparticle dispersion relations vary with the direction of the momentum, yielding gaps which range from zero up to a maximum of Δ . The condensate is dominated by those “ring-shaped” regions in momentum space in which a quark pair with total momentum $2\mathbf{q}$ has both members of the pair within approximately Δ of their respective Fermi surfaces.

The gap equation which determines Δ was derived in Ref. [10] using variational methods, along the lines of Refs. [25,31]. It has since been rederived using now more familiar methods, namely a diagrammatic derivation of the one-loop Schwinger-Dyson equation with a Nambu-Gorkov propagator modified to describe the spatially varying condensate [29]. This gap equation can then be used to show that crystalline color superconductivity is favored over no ud pairing for $\delta\mu < \delta\mu_2$. If quarks interact via a weak point-like four-fermion interaction, $\delta\mu_2 \approx 0.754\Delta_0$ [24,25,31,10]. For a stronger four-fermion interaction, $\delta\mu_2$ tends to decrease, closing the crystalline color superconductivity window [10].

Crystalline color superconductivity is favored for $\delta\mu_1 < \delta\mu < \delta\mu_2$. As $\delta\mu$ increases, one finds a first order phase transition from the ordinary BCS phase to the crystalline color superconducting phase at $\delta\mu = \delta\mu_1$ and then a second order phase transition at $\delta\mu = \delta\mu_2$ at which Δ decreases to zero. Because the condensation energy in the LOFF phase is much smaller than that of the BCS condensate at $\delta\mu = 0$, the value of $\delta\mu_1$ is almost identical to that at which the naive unpairing transition from the BCS state to the state with no pairing would occur if one ignored the possibility of a LOFF phase, namely $\delta\mu_1 = \Delta_0/\sqrt{2}$. For all practical purposes, therefore, the LOFF gap equation is not required in order to determine $\delta\mu_1$. The LOFF gap equation is used to determine $\delta\mu_2$ and the properties of the crystalline color superconducting phase [10].

In this paper, we generalize the diagrammatic analysis of Ref. [29] to analyze the crystalline color superconducting phase which occurs when quarks interact by the exchange of a propagating gluon, as is quantitatively valid at asymptotically high densities. At weak coupling, quark-quark scattering by single-gluon exchange is dominated by forward scattering. In most scatterings, the angular positions of the quarks on their respective Fermi surfaces do not change much. As a consequence, the weaker the coupling the more the physics can be thought of as a sum of many $(1+1)$ -dimensional theories, with only rare large-angle scatterings able to connect one direction in momentum space with others [17]. Suppose for a moment that we were analyzing a truly $(1+1)$ -dimensional theory. The momentum-space geometry of the LOFF state in one spatial dimension is qualitatively different from that in three. Instead of Fermi surfaces, we would have only “Fermi points” at $\pm\mu_u$ and $\pm\mu_d$. The only choice of $|\mathbf{q}|$ which allows pairing between u and d quarks at their respective Fermi points is $|\mathbf{q}| = \delta\mu$. In $(3+1)$ dimensions, in contrast, $|\mathbf{q}| > \delta\mu$ is favored because it allows LOFF pairing in “ring-shaped” regions of the Fermi surface, rather than just at antipodal points [24,25,10]. Also, in striking contrast to the $(3+1)$ -dimensional case, it has long

been known that in a true $(1+1)$ -dimensional theory with a point-like interaction between fermions, $\delta\mu_2/\Delta_0 \rightarrow \infty$ in the weak-interaction limit [32].

We therefore expect that in $(3+1)$ -dimensional QCD with the interaction given by single-gluon exchange, as $\bar{\mu} \rightarrow \infty$ and $g(\bar{\mu}) \rightarrow 0$ the $|\mathbf{q}|$ which characterizes the LOFF phase should become closer and closer to $\delta\mu$ and $\delta\mu_2/\Delta_0$ should diverge. We shall demonstrate both effects, and shall furthermore show that both are clearly in evidence already at the rather large coupling $g = 3.43$, corresponding to $\bar{\mu} = 400$ MeV using the conventions of Refs. [19,23]. At this coupling, $\delta\mu_2/\Delta_0 \approx 1.2$, meaning that $(\delta\mu_2 - \delta\mu_1) \approx (1.2 - 1/\sqrt{2})\Delta_0$, which is much larger than $(0.754 - 1/\sqrt{2})\Delta_0$. If we go to much higher densities, where the calculation is under quantitative control, we find an even more striking enhancement: when $g = 0.79$ we find $\delta\mu_2/\Delta_0 > 1000$! We see that (relative to expectations based on experience with point-like interactions) the crystalline color superconductivity window is wider by more than four orders of magnitude at this weak coupling, and is about one order of magnitude wider at accessible densities if weak-coupling results are applied there.

In Section II, we present the gap equation derived from a Schwinger-Dyson equation as in Ref. [29], but now assuming that quarks interact via single-gluon exchange. As our goal is the evaluation of $\delta\mu_2$, the value of $\delta\mu$ at which the crystalline color superconducting gap vanishes, in Section III we take the $\Delta \rightarrow 0$ limit of the gap equation. Analysis of the resulting equation yields $\delta\mu_2$. In Section IV we look ahead to implications of our central result that the crystalline color superconductivity window is wider by (an) order(s) of magnitude than previously thought.

II. THE GAP EQUATION FOR CRYSTALLINE COLOR SUPERCONDUCTIVITY AT WEAK COUPLING

In the crystalline color superconducting phase [24,25,10,29], the condensate contains pairs of u and d quarks with momenta such that the total momentum of each Cooper pair is given by $2\mathbf{q}$, with the direction of \mathbf{q} chosen spontaneously. Such a condensate varies periodically in space with wavelength $\pi/|\mathbf{q}|$ as in (2). Wherever there is an instability towards (2), we expect that the true ground state will be a crystalline condensate which varies in space like a sum of several such plane waves with the same $|\mathbf{q}|$. As in Refs. [10,29], we focus here on finding the region of $\delta\mu$ where an instability towards (2) occurs, leaving the determination of the favored crystal structure to future work.

We begin by reviewing the field theoretic framework for the analysis of crystalline color superconductivity presented in Ref. [29]. In order to describe pairing between u quarks with momentum $\mathbf{p} + \mathbf{q}$ and d quarks with momentum $-\mathbf{p} + \mathbf{q}$, we must use a modified Nambu-Gorkov spinor defined as

$$\Psi(p, q) = \begin{pmatrix} \psi_u(p + q) \\ \psi_d(p - q) \\ \bar{\psi}_d^T(-p + q) \\ \bar{\psi}_u^T(-p - q) \end{pmatrix}. \quad (3)$$

Note that by q we mean the four-vector $(0, \mathbf{q})$. The Cooper pairs have nonzero total momentum, and the ground state condensate (2) is static. The momentum dependence of (3)

is motivated by the fact that in the presence of a crystalline color superconducting condensate, anomalous propagation does not only mean picking up or losing two quarks from the condensate. It also means picking up or losing momentum $2\mathbf{q}$. The basis (3) has been chosen so that the inverse fermion propagator in the crystalline color superconducting phase is diagonal in p -space and is given by

$$S^{-1}(p, q) = \begin{bmatrix} \not{p} + \not{q} + \mu_u \gamma_0 & 0 & -\bar{\Delta}(p, -q) & 0 \\ 0 & \not{p} - \not{q} + \mu_d \gamma_0 & 0 & \bar{\Delta}(p, q) \\ -\Delta(p, -q) & 0 & (\not{p} - \not{q} - \mu_d \gamma_0)^T & 0 \\ 0 & \Delta(p, q) & 0 & (\not{p} + \not{q} - \mu_u \gamma_0)^T \end{bmatrix}, \quad (4)$$

where $\bar{\Delta} = \gamma_0 \Delta^\dagger \gamma_0$ and $\Delta = C \gamma_5 \epsilon^{\alpha\beta 3}$ is a matrix with color (α, β) and Dirac indices suppressed. Note that the condensate is explicitly antisymmetric in flavor. $2\mathbf{p}$ is the relative momentum of the quarks in a given pair and is different for different pairs. In the gap equation below, we shall integrate over p_0 and \mathbf{p} . As desired, the off-diagonal blocks describe anomalous propagation in the presence of a condensate of diquarks with momentum $2\mathbf{q}$. The choice of basis we have made is analogous to that introduced previously in the analysis of a crystalline quark-antiquark condensate [33].

We obtain the gap equation by solving the one-loop Schwinger-Dyson equation given by

$$S^{-1}(k, q) - S_0^{-1}(k, q) = -g^2 \int \frac{d^4 p}{(2\pi)^4} \Gamma_\mu^A S(p, q) \Gamma_\nu^B D_{AB}^{\mu\nu}(k - p), \quad (5)$$

where $D_{AB}^{\mu\nu} = D^{\mu\nu} \delta_{AB}$ is the gluon propagator, S is the full quark propagator, whose inverse is given by (4), and S_0 is the fermion propagator in the absence of interaction, given by S with $\Delta = 0$. The vertices are defined as follows:

$$\Gamma_\mu^A = \begin{pmatrix} \gamma_\mu \lambda^A / 2 & 0 & 0 & 0 \\ 0 & \gamma_\mu \lambda^A / 2 & 0 & 0 \\ 0 & 0 & -(\gamma_\mu \lambda^A / 2)^T & 0 \\ 0 & 0 & 0 & -(\gamma_\mu \lambda^A / 2)^T \end{pmatrix}. \quad (6)$$

In previous analyses [10,29], a point-like interaction was introduced by replacing $g^2 D^{\mu\nu}$ by $g^{\mu\nu}$ times a constant. Here, instead, we analyze the interaction between quarks given by the exchange of a medium-modified gluon and thus use a gluon propagator given by

$$D_{\mu\nu}(p) = \frac{P_{\mu\nu}^T}{p^2 - G(p)} + \frac{P_{\mu\nu}^L}{p^2 - F(p)} - \xi \frac{p_\mu p_\nu}{p^4}, \quad (7)$$

where ξ is the gauge parameter, $G(p)$ and $F(p)$ are functions of p_0 and $|\mathbf{p}|$, and the projectors $P_{\mu\nu}^{T,L}$ are defined as follows:

$$P_{ij}^T = \delta_{ij} - \hat{p}_i \hat{p}_j, \quad P_{00}^T = P_{0i}^T = 0, \quad P_{\mu\nu}^L = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} - P_{\mu\nu}^T. \quad (8)$$

The functions F and G describe the effects of the medium on the gluon propagator. If we neglect the Meissner effect (that is, if we neglect the modification of $F(p)$ and $G(p)$ due to the gap Δ in the fermion propagator) then $F(p)$ describes Thomas-Fermi screening and $G(p)$

describes Landau damping and they are given in the hard dense loop (HDL) approximation by [34]

$$\begin{aligned} F(p) &= m^2 \frac{p^2}{|\mathbf{p}|^2} \left(1 - \frac{ip_0}{|\mathbf{p}|} Q_0 \left(\frac{ip_0}{|\mathbf{p}|} \right) \right), \quad \text{with } Q_0(x) = \frac{1}{2} \log \left(\frac{x+1}{x-1} \right), \\ G(p) &= \frac{1}{2} m^2 \frac{ip_0}{|\mathbf{p}|} \left[\left(1 - \left(\frac{ip_0}{|\mathbf{p}|} \right)^2 \right) Q_0 \left(\frac{ip_0}{|\mathbf{p}|} \right) + \frac{ip_0}{|\mathbf{p}|} \right], \end{aligned} \quad (9)$$

where $m^2 = g^2 \mu^2 / \pi^2$ is the Debye mass for $N_f = 2$.

The method of solving the Schwinger-Dyson equation has been outlined in Ref. [29]. One obtains coupled integral equations for the gap parameters Δ_1 , which describes particle-particle and hole-hole pairing, and $\Delta_{2,3,4}$, which describe particle-antiparticle and antiparticle-antiparticle pairing. Because only Δ_1 describes pairing of quarks near their respective Fermi surfaces, this is the gap parameter which describes the gap in the quasiparticle spectrum and is therefore the gap parameter of physical interest. For the same reason, the gap equation is dominated by the terms containing Δ_1 . Upon dropping $\Delta_{2,3,4}$ and renaming $\Delta_1 \rightarrow \Delta$, the gap equation derived in Ref. [29] can be written as:

$$\begin{aligned} \Delta(k_0) &= \frac{-ig^2}{3 \sin^2 \frac{\beta(k,k)}{2}} \int \frac{d^4 p}{(2\pi)^4} \frac{\Delta(p_0)}{(p_0 + E_1)(p_0 - E_2)} \\ &\quad \times \left[\frac{C_F}{(k-p)^2 - F(k-p)} + \frac{C_G}{(k-p)^2 - G(k-p)} + \frac{C_\xi \xi}{(k-p)^2} \right], \end{aligned} \quad (10)$$

where

$$\begin{aligned} C_F &= \cos^2 \frac{\beta(k,p)}{2} \cos^2 \frac{\beta(p,k)}{2} - \cos^2 \frac{\beta(k,-p)}{2} \cos^2 \frac{\beta(-p,k)}{2} - \sin^2 \frac{\beta(k,k)}{2} \sin^2 \frac{\beta(p,p)}{2}, \\ C_G &= \frac{\cos \beta(k,-p) \cos \beta(-p,k) - \cos \beta(k,p) \cos \beta(p,k)}{2} - 2 \sin^2 \frac{\beta(k,k)}{2} \sin^2 \frac{\beta(p,p)}{2} \\ &\quad - \cos \alpha(k,p) \left(\cos \alpha(p,k) \sin^2 \frac{\beta(k,-p)}{2} + \cos \alpha(-p,-k) \sin^2 \frac{\beta(k,p)}{2} \right) \\ &\quad - \cos \alpha(-k,-p) \left(\cos \alpha(p,k) \sin^2 \frac{\beta(p,k)}{2} + \cos \alpha(-p,-k) \sin^2 \frac{\beta(-p,k)}{2} \right), \\ C_\xi &= \sin^2 \frac{\beta(k,p)}{2} \sin^2 \frac{\beta(p,k)}{2} - \sin^2 \frac{\beta(k,k)}{2} \sin^2 \frac{\beta(p,p)}{2} - \sin^2 \frac{\beta(k,-p)}{2} \sin^2 \frac{\beta(-p,k)}{2} \\ &\quad + \cos \alpha(k,p) \left(\cos \alpha(p,k) \sin^2 \frac{\beta(k,-p)}{2} + \cos \alpha(-p,-k) \sin^2 \frac{\beta(k,p)}{2} \right) \\ &\quad + \cos \alpha(-k,-p) \left(\cos \alpha(p,k) \sin^2 \frac{\beta(p,k)}{2} + \cos \alpha(-p,-k) \sin^2 \frac{\beta(-p,k)}{2} \right), \end{aligned} \quad (11)$$

and

$$\begin{aligned} E_1(\mathbf{p}) &= +\delta\mu + \frac{1}{2} (|\mathbf{p} + \mathbf{q}| - |\mathbf{p} - \mathbf{q}|) + \frac{1}{2} \sqrt{(|\mathbf{p} + \mathbf{q}| + |\mathbf{p} - \mathbf{q}| - 2\bar{\mu})^2 + 4\Delta^2 \sin^2 \frac{\beta(p,p)}{2}}, \\ E_2(\mathbf{p}) &= -\delta\mu - \frac{1}{2} (|\mathbf{p} + \mathbf{q}| - |\mathbf{p} - \mathbf{q}|) + \frac{1}{2} \sqrt{(|\mathbf{p} + \mathbf{q}| + |\mathbf{p} - \mathbf{q}| - 2\bar{\mu})^2 + 4\Delta^2 \sin^2 \frac{\beta(p,p)}{2}}, \end{aligned} \quad (12)$$

with

$$\begin{aligned}\cos \alpha(k, p) &= (\widehat{k - q}) \cdot (\widehat{k - p}) , \\ \cos \beta(k, p) &= (\widehat{q + k}) \cdot (\widehat{q - p}) .\end{aligned}\tag{13}$$

In the next section, we shall use this gap equation to obtain $\delta\mu_2$, the upper boundary of the crystalline color superconductivity window.

III. THE ZERO-GAP CURVE AND THE CALCULATION OF $\delta\mu_2$

Solving the full gap equation, Eq. (10), is numerically challenging. Fortunately, our task is simpler. As $\delta\mu \rightarrow \delta\mu_2$, the gap $\Delta \rightarrow 0$. Therefore, in order to determine $\delta\mu_2$ we need only analyze the $\Delta \rightarrow 0$ limit of Eq. (10). Upon taking this limit, we shall find a “zero-gap curve” relating $\delta\mu$ and $|\mathbf{q}|$, as obtained for a point-like interaction in Refs. [25,31,10]. The largest value of $\delta\mu$ on this curve is $\delta\mu_2$, and the $|\mathbf{q}|$ at this point on the curve is the favored value of $|\mathbf{q}|$ in the crystalline color superconducting phase for $\delta\mu \rightarrow \delta\mu_2$.

To take the $\Delta \rightarrow 0$ limit, we first divide both sides of Eq. (10) by $\Delta(k_0)$. We must be careful, however: simply cancelling the $\Delta(p_0)/\Delta(k_0)$ which now occurs on the right-hand side would yield an integral which diverges in the ultraviolet. This divergence is in fact regulated by the p_0 -dependence of $\Delta(p_0)$ itself. While the precise shape of $\Delta(p_0)$ remains unknown unless we solve the full gap equation (10), we assume $\Delta(p_0)$ has a similar p_0 dependence as in the BCS case [19,23], where it is relatively flat for $p_0 < \mu$ and then decreases rapidly to zero as p_0 increases to several times μ . This rapid decrease regulates the integral. This means that for $k_0 \ll \bar{\mu}$, we can approximate $\Delta(p_0)/\Delta(k_0)$ by a step function which imposes an ultraviolet cutoff on the p_0 integral at $p_0 = \Lambda$, where Λ is of order several times $\bar{\mu}$. We shall investigate the dependence of our results on the choice of Λ . After the factor $\Delta(p_0)/\Delta(k_0)$ has been replaced by a cutoff, it is safe to simply set Δ to zero everywhere else in the integrand.

After rotating to Euclidean space we obtain

$$\begin{aligned}1 &= \frac{g^2}{3 \sin^2 \frac{\beta(k,k)}{2}} \int^\Lambda \frac{d^4 p}{(2\pi)^4} \frac{1}{(p_0 - iE_1)(p_0 + iE_2)} \\ &\times \left[\frac{C_F}{(k-p)^2 + F(k-p)} + \frac{C_G}{(k-p)^2 + G(k-p)} + \frac{C_\xi \xi}{(k-p)^2} \right],\end{aligned}\tag{14}$$

where the energies $E_{1,2}$ can be obtained from Eq. (12) by setting Δ equal to 0. We also simplify the functions F and G to

$$F(p) = m^2, \quad G(p) = \frac{\pi}{4} m^2 \frac{p_0}{|\mathbf{p}|},\tag{15}$$

valid for $p_0 \ll |\mathbf{p}| \sim \bar{\mu}$ and to leading order in perturbation theory. We shall work in the gauge with $\xi = 0$, as this will allow us to compare our results for $\delta\mu_2$ to values of Δ_0 obtained in the same gauge in the analysis of the $\delta\mu = |\mathbf{q}| = 0$ version of Eq. (10) in Ref. [19]. As the gauge dependence in the calculation of Δ_0 decreases as g is reduced below ~ 1 [23], we expect that the same is true here for $\delta\mu_2$. We leave an analysis of the ξ -dependence

of $\delta\mu_2$ to the future. In evaluating the right-hand side of Eq. (14), we should set $k_0 \sim \Delta$, but instead choose $k_0 = 0$ for simplicity as we expect $\Delta(k_0)$ to be almost constant for $k_0 < \Delta$ [19,23]. We must choose \mathbf{k} in the region of momentum space which dominates the pairing: for $\delta\mu = |\mathbf{q}| = 0$, for example, one takes $|\mathbf{k}| = \mu$ [19,23]. Now, we choose \mathbf{k} on the ring in \mathbf{k} -space which describes pairs of quarks both of whose momenta ($\mathbf{k} + \mathbf{q}$ and $\mathbf{k} - \mathbf{q}$) lie on their respective noninteracting Fermi surfaces, as this is where pairing is most important [24,25,10]. For $|\mathbf{q}| = \delta\mu \neq 0$, as opposed to $|\mathbf{q}| > \delta\mu \neq 0$, the ring degenerates to the point in momentum space with $|\mathbf{k}| = \bar{\mu}$ and \mathbf{k} antiparallel to \mathbf{q} .

We do the p_0 integral analytically. There are six poles in the complex p_0 plane: two from the quark propagator located at iE_1 at $-iE_2$ and four from the gluon propagator located at $\pm i\sqrt{|\mathbf{k} - \mathbf{p}|^2 + m^2}$, $-i(\pi m^2 \pm \sqrt{\pi^2 m^4 + 64|\mathbf{k} - \mathbf{p}|^4})/(8|\mathbf{k} - \mathbf{p}|)$. The poles from the gluon propagator have residues which are smaller than those from the quark propagator by factors of $\bar{\mu}$. We therefore keep only the poles at iE_1 and $-iE_2$. Furthermore, we can drop the p_0^2 pieces in the gluon propagators because at the $E_{1,2}$ poles they are negligible. Upon doing the contour integral over p_0 , we notice that we only obtain a non-zero result if both E_1 and E_2 are positive. This defines the “pairing region” \mathcal{P} of Refs. [10,29]. We get

$$1 = \frac{g^2}{3 \sin^2 \frac{\beta(k,k)}{2}} \int_{\mathcal{P}}^{\Lambda} \frac{p^2 dp d\Omega}{(2\pi)^3} \frac{1}{(E_1 + E_2)} \left(\frac{C_F}{|\mathbf{k} - \mathbf{p}|^2 + F} + \frac{C_G}{|\mathbf{k} - \mathbf{p}|^2 + G} \right), \quad (16)$$

where the integral is taken over the pairing region \mathcal{P} with an upper cutoff on the p integral of Λ . The remaining integrals are done numerically.

Upon doing the integrals, Eq. (16) becomes an equation relating $|\mathbf{q}|$ and $\delta\mu$. We explore the $(|\mathbf{q}|, \delta\mu)$ plane, evaluating the right-hand side of Eq. (16) at each point, and in so doing map out the “zero-gap curves” along which Eq. (16) is satisfied. Several zero-gap curves are shown in Fig. 1, in order to exhibit the dependence of the curve on the coupling g and on the parameter Λ . The top panel is for $g = 3.4328$: if we use one-loop running with $\Lambda_{\text{QCD}} = 200$ MeV and $N_f = 2$, this corresponds to $g(\bar{\mu})$ for $\bar{\mu} = 400$ MeV. The lower panel is for $g = 2.2528$ corresponding in the same sense to $\bar{\mu} = 10^3$ MeV. The three curves in each plot are drawn with $\Lambda/\bar{\mu} = 2.0, 2.5$, and 3.0 . The choice of Λ makes some difference to the scale of the zero-gap curves, but does not change the qualitative behavior. We are only interested in the qualitative behavior, since the couplings g which are appropriate at accessible densities are much too large for the analysis to be under quantitative control anyway. In subsequent calculations, we shall take $\Lambda/\bar{\mu} = 2.5$ as our canonical choice. Removing the artificially introduced parameter Λ would require solving the full gap equation (10).

We only show the zero-gap curve for $|\mathbf{q}| \geq \delta\mu$, because the favored value of $|\mathbf{q}|$ must lie in this region [24,25,31,10]. Each zero-gap curve in Fig. 1 begins somewhere on the line $\delta\mu = |\mathbf{q}|$, moves up and to the right following the line $\delta\mu = |\mathbf{q}|$ very closely, and then turns around sharply and heads downward, eventually reaching the $\delta\mu = 0$ axis. This behavior is most clearly seen in the upper panel. The smaller the value of g , the more sharply the curve turns around and the more closely the downward-going curve hugs the $\delta\mu = |\mathbf{q}|$ line. Solutions to the full gap equation (10) with $\Delta \neq 0$ occur in the region bounded by the zero-gap curve, between it and the $\delta\mu = |\mathbf{q}|$ line. As g is decreased, this region becomes a sharper and sharper sliver, squeezed closer and closer to the $\delta\mu = |\mathbf{q}|$ line. Since in $(1+1)$ dimensions $|\mathbf{q}| = \delta\mu$ is favored, the change in the zero-gap curves from the upper panel to

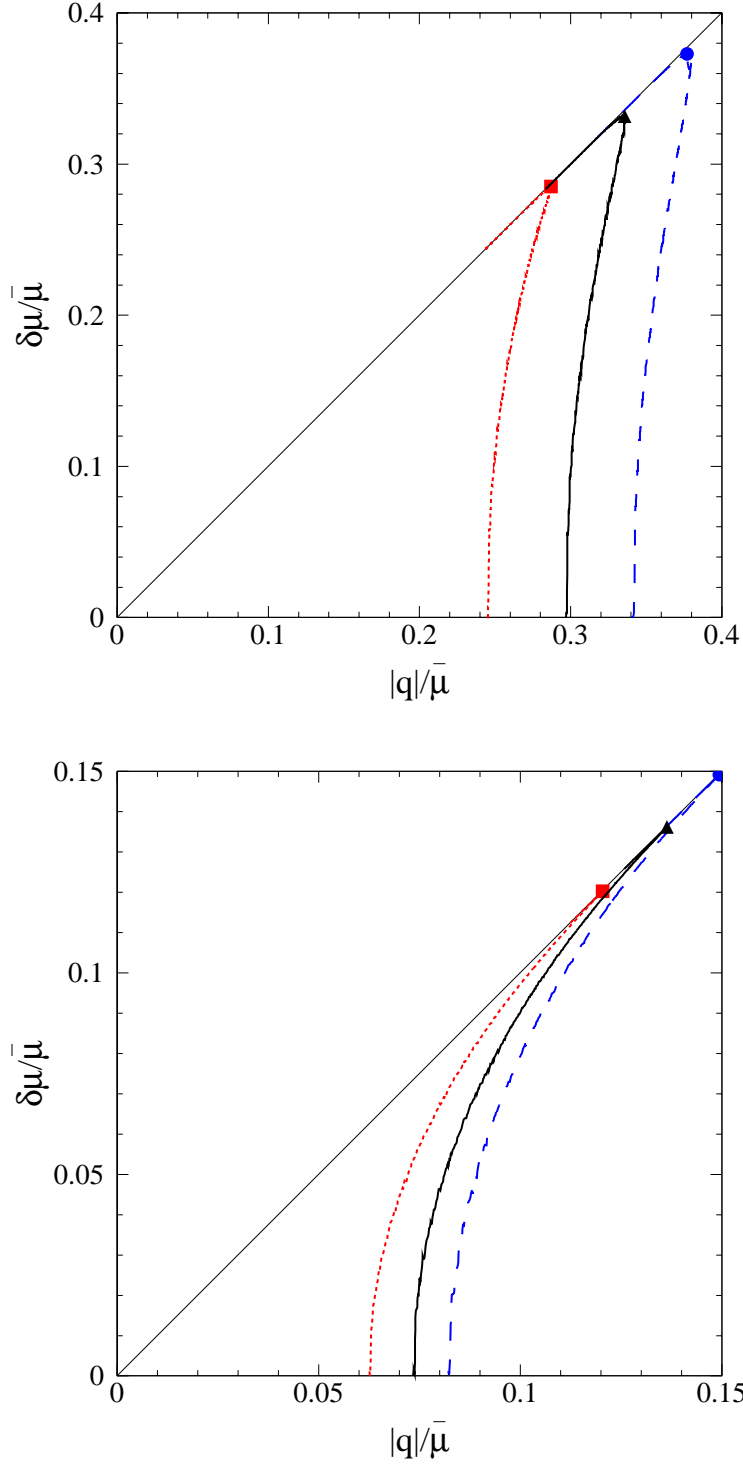


FIG. 1. Zero-gap curves for different choices of the cutoff Λ . The upper panel is for $g = 3.4328$, corresponding to $\bar{\mu} = 400$ MeV, while the lower panel is for $g = 2.2428$, corresponding to $\bar{\mu} = 10^3$ MeV. The three curves in each panel correspond to $\Lambda/\bar{\mu} = 2.0$ (dotted red), 2.5 (solid black), and 3.0 (dashed blue). The square, triangle, and circle mark the locations of $\delta\mu_2$ for each of the three curves.

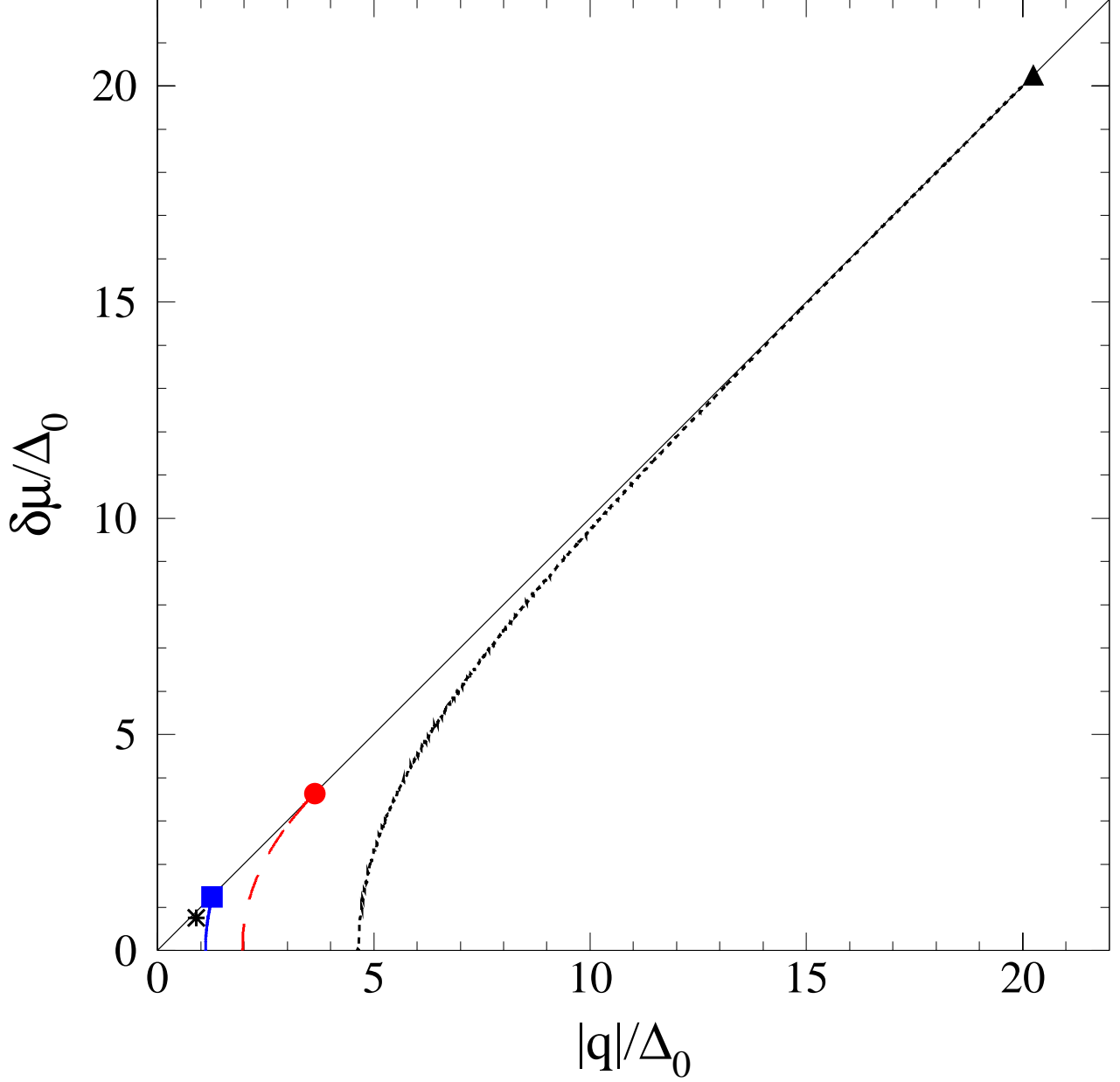


FIG. 2. Zero-gap curves for different choices of $\bar{\mu}$. Each curve is normalized to the BCS gap Δ_0 for the corresponding $\bar{\mu}$. The three curves are $\bar{\mu} = 400$ MeV (solid blue), 10^3 MeV (dashed red), and 10^4 MeV (dotted black). Again, $\delta\mu_2$ is marked for each curve. For comparison, the star is the position of $\delta\mu_2$ for a weak point-like interaction.

the lower vividly demonstrates how the physics embodied in the gap equation (10) becomes effectively (1 + 1)-dimensional as g is reduced.

The largest value of $\delta\mu$ on the zero-gap curve is $\delta\mu_2$. For $\delta\mu < \delta\mu_2$, a range of values of $|\mathbf{q}|$ can be found for which a crystalline color superconducting phase with $\Delta \neq 0$ exists. The $|\mathbf{q}|$ within this range which results in a phase with the lowest free energy is favored, and

$\bar{\mu}$ (MeV)	g	$\delta\mu_2/\bar{\mu}$	$ \mathbf{q} /\delta\mu_2$	Δ_0 (MeV)	$\delta\mu_2/\Delta_0$
400	3.4328	0.3317	1.012	107	1.24
10^3	2.2528	0.1362	1.001	37.5	3.63
10^4	1.4450	0.0241	1.000	11.9	20.3
10^5	1.1464	5.23×10^{-3}	1.0000	8.0	65.3
10^6	0.9793	1.71×10^{-3}	1.0000	7.63	225
10^7	0.8689	5.84×10^{-4}	1.0000	9.16	638
10^8	0.7890	1.63×10^{-4}	1.0000	13.0	1254

TABLE I. Positions of $\delta\mu_2$ and corresponding $|\mathbf{q}|$ for seven values of g : if we use one-loop running with $\Lambda_{\text{QCD}} = 200$ MeV, these correspond to $g(\bar{\mu})$ for the values of $\bar{\mu}$ shown. For each g , we give $\delta\mu_2/\bar{\mu}$ and $|\mathbf{q}|/\delta\mu_2$ as determined from the zero-gap curves we calculate. To obtain $\delta\mu_2/\Delta_0$, we take Δ_0 from Ref. [19].

the corresponding Δ obtained by solving Eq. (10) at that $|\mathbf{q}|$ is favored. At $\delta\mu_2$, however, $\Delta \rightarrow 0$. For each curve in Fig. 1, the location of $\delta\mu_2$ is marked. In all cases, $|\mathbf{q}|$ at $\delta\mu_2$ is very close to $\delta\mu_2$.

We see from Fig. 1 that $\delta\mu_2/\bar{\mu}$ decreases as g decreases, but this is not the important comparison. Recall that the crystalline color superconducting phase occurs within a window $\delta\mu_1 < \delta\mu < \delta\mu_2$ where $\delta\mu_1 = \Delta_0/\sqrt{2}$ at weak coupling, with Δ_0 the BCS gap obtained at $\delta\mu = 0$. To evaluate the width of the window, we must therefore also compare $\delta\mu_2$ to Δ_0 . We obtain Δ_0 from Ref. [19], and in Fig. 2 we plot the zero-gap curves for $\bar{\mu}$ equal to 400 MeV, 10^3 MeV, and 10^4 MeV, in each case scaled by the corresponding BCS gap Δ_0 . In contrast, the star marks the location of $\delta\mu_2$ obtained in a theory with a weak point-like interaction: $\delta\mu_2 = 0.754\Delta_0$ and $|\mathbf{q}| = 0.906\Delta_0$.

The effect of the single-gluon exchange interaction, then, is to increase $\delta\mu_2/\Delta_0$ dramatically. Because this interaction favors forward scattering at weak coupling, it causes $\delta\mu_2$ to approach the line $|\mathbf{q}| = \delta\mu$ and $\delta\mu_2/\Delta_0$ to diverge, as occurs in $(1+1)$ dimensions where there are only “Fermi points”. In Table I, we show the $\delta\mu_2/\Delta_0$ values for different values of $\bar{\mu}$, with all calculations done for $\Lambda/\bar{\mu} = 2.5$. Already at $g = 3.4328$, corresponding to $\bar{\mu} = 400$ MeV, $\delta\mu_2$ is within 1% of the $|\mathbf{q}| = \delta\mu$ line and $\delta\mu_2 = 1.24\Delta_0$. Although this result will depend somewhat on Λ , it means that the crystalline color superconducting window is about ten times wider than for a point-like interaction. Once $g = 1.445$, corresponding to $\bar{\mu} = 10^4$ MeV, $\delta\mu_2$ is closer than one part in a thousand to the $|\mathbf{q}| = \delta\mu$ line and $\delta\mu_2 = 20.3\Delta_0$, meaning that the crystalline color superconducting window is about four hundred times wider than for a point-like interaction.

IV. IMPLICATIONS AND FUTURE WORK

We have found that $\delta\mu_2/\Delta_0$ diverges in QCD as the weak-coupling, high-density limit is taken. Applying results valid at asymptotically high densities to those of interest in compact stars, namely $\bar{\mu} \sim 400$ MeV, we find that even here the crystalline color superconductivity

window is an order of magnitude wider than that obtained previously upon approximating the interaction between quarks as point-like.

This discovery has significant implications for the QCD phase diagram and may have significant implications for compact stars. At high enough baryon density the CFL phase in which all quarks pair to form a spatially uniform BCS condensate is favored. Suppose that as the density is lowered the nonzero strange quark mass induces the formation of some less symmetrically paired quark matter before the density is lowered so much that baryonic matter is obtained. In this less symmetric quark matter, some quarks may yet form a BCS condensate. Those which do not, however, will have differing Fermi momenta. These will form a crystalline color superconducting phase if the differences between their Fermi momenta lie within the appropriate window. In QCD, the interaction between quarks is forward-scattering dominated and the crystalline color superconductivity window is consequently wide open. This phase is therefore generic, occurring almost anywhere there are some quarks which cannot form BCS pairs. Evaluating the critical temperature T_c above which the crystalline condensate melts requires solving the nonzero temperature gap equation obtained from Eq. (10) as described in Ref. [29], but we expect that all but the very youngest compact stars are colder than T_c . This suggests that wherever quark matter which is not in the CFL phase occurs within a compact star, rotational vortices may be pinned resulting in the generation of glitches as the star spins down.

Solidifying the implications of our results requires further work in several directions. First, we must confirm that pushing Fermi surfaces apart via quark mass differences has the same effect as pushing them apart via a $\delta\mu$ introduced by hand. Second, we must extend the analysis to the three flavor theory of interest. Third, we need to re-evaluate $\delta\mu_2$ by comparing the crystalline color superconducting phase to a BCS phase in which spin-1 pairing between quarks of the same flavor is allowed [35]. The results of Ref. [10] suggest that this can be neglected, but confirming this in the present case requires solving the full gap equation (10). And, fourth, before evaluating the pinning force on a rotational vortex and making predictions for glitch phenomena, we need to understand which crystal structure is favored.

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